Thetafunktionen positiv definiter quadratischer Formen III

Thetafunktionen mit Charakteristik

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Verwendete Literatur: S. ZWEGERS, *Lecture Notes on Modular Forms*

Wiederholung

Definition (Gerade Matrix)

$r \in \mathbb{N}$; $A \in M_{r,r}(\mathbb{Z})$ symmetrisch $\rightarrow$
A gerade $:\Leftrightarrow \forall i \in \{1, \ldots, r\} : A_{ii}$ gerade

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Definition (Peterssonscher Strichoperator)

$k \in \mathbb{Z}$; $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_{2,2}(\mathbb{R})$; $f : \mathbb{H} \rightarrow \mathbb{C}$ holomorph $\rightarrow$

$$(f|_k(\gamma))(\tau) := \frac{\det(\gamma)^{\frac{k}{2}}}{(c\tau + d)^k} f(\gamma.\tau) \quad \text{mit} \quad \gamma.\tau := \frac{a\tau + b}{c\tau + d}$$

Definition (A-assoziierte Formen)

$r \in \mathbb{N}$; $A \in M_{r,r}(\mathbb{Z})$ gerade $\rightarrow$
Assoziierte quadratische Form: $Q : \mathbb{R}^r \rightarrow \mathbb{R}, Q(x) := \frac{1}{2}x^T Ax$
Assoziierte Bilinearform: $Q : \mathbb{R}^r \times \mathbb{R}^r \rightarrow \mathbb{R}, B(x,y) := x^T Ay$

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Definition (A-assoziierte Thetafunktion)

$r \in \mathbb{N}$; $A \in M_{r,r}(\mathbb{Z})$ gerade, positiv definit $\rightarrow$

$$\forall \tau \in \mathbb{H} : q := \exp(2\pi i\tau) : \Theta_A(\tau) := \sum_{n \in \mathbb{Z}^r} q^{Q(n)}$$

Definition

Definition (A-assoziierte Thetafunktion mit Charakteristik λ)

$r \in \mathbb{N}$; $A \in M_{r,r}(\mathbb{Z})$ gerade, positiv definit; $\lambda \in A^{-1}\mathbb{Z}^r \rightarrow$

$$\forall \tau \in \mathbb{H} : q := \exp(2\pi i \tau) : \Theta_{A,\lambda}(\tau) := \sum_{n \in \lambda + \mathbb{Z}^r} q^{Q(n)}$$

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Beweis.

$$\Theta_{A,0}(\tau) = \sum_{n \in 0+\mathbb{Z}^r} q^{Q(n)} = \sum_{n \in \mathbb{Z}^r} q^{Q(n)} = \Theta_A(\tau) \qquad \square$$

Bemerkung (Reduktion von λ)

$$\forall \mu \in \mathbb{Z}^r : \Theta_{A,\lambda+\mu} = \Theta_{A,\lambda}$$

$\rightarrow$ Es reicht $\lambda \in A^{-1}\mathbb{Z}^r/\mathbb{Z}^r$ zu betrachten.

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$\mu + \mathbb{Z}^r = \mathbb{Z}^r \Rightarrow$

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Bemerkung (Lineare Abhängigkeit)

$$\forall \lambda \in A^{-1}\mathbb{Z}^r/\mathbb{Z}^r : \Theta_{A,-\lambda} = \Theta_{A,\lambda}$$

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Beweis.

$\forall x \in \mathbb{R}^r : Q(-x) = Q(x) \wedge -\mathbb{Z}^r = \mathbb{Z}^r \Rightarrow$

$$\Theta_{A,-\lambda}(\tau) = \sum_{n \in -\lambda + \mathbb{Z}^r} q^{Q(n)} = \sum_{n \in -(-\lambda + \mathbb{Z}^r) = \lambda - \mathbb{Z}^r} q^{Q(-n)}$$

$$= \sum_{n \in \lambda + \mathbb{Z}^r} q^{Q(-n)} = \sum_{n \in \lambda + \mathbb{Z}^r} q^{Q(n)} = \Theta_{A,\lambda}(\tau) \qquad \square$$

Transformationsformeln

Theorem (Transformationsformeln)

$r \in \mathbb{N}$; $A \in M_{r,r}(\mathbb{Z})$ gerade, positiv definit; $\lambda \in A^{-1}\mathbb{Z}^r \Rightarrow$

$$\Theta_{A,\lambda}(\tau+1) = \exp(2\pi i Q(\lambda))\Theta_{A,\lambda}(\tau) \qquad \text{(i)}$$

$$\Theta_{A,\lambda}\left(-\frac{1}{\tau}\right) = \frac{(-i\tau)^{\frac{r}{2}}}{\sqrt{\det(A)}} \sum_{\mu \in A^{-1}\mathbb{Z}^r/\mathbb{Z}^r} \exp(2\pi i B(\lambda,\mu))\Theta_{A,\mu}(\tau) \qquad \text{(ii)}$$

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Lemma (1)

$\star_{Theorem} \Rightarrow f(x) := \exp\left(-\frac{2\pi Q(x)}{t}\right) \rightarrow$

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Lemma (2)

$$f \in \mathcal{S}(\mathbb{R}^r) \Rightarrow \sum_{n \in \mathbb{Z}^r} f(n) = \sum_{n \in \mathbb{Z}^r} (\mathcal{F}f)(n)$$

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&= \exp(2\pi i\lambda x)(\mathcal{F}f)(x) && \Big| \quad \text{Lemma}(1) \\
&= \frac{t^{\frac{r}{2}}}{\sqrt{\det(A)}}\exp(-2\pi tQ^{\star}(x) + 2\pi i\lambda x)
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$|$ linkslinear, $\forall \gamma_1, \gamma_2 \in \Gamma_1 : (\Theta_{A,\lambda}|_{\frac{r}{2}}(\gamma_1))|_{\frac{r}{2}}(\gamma_2) = \Theta_{A,\lambda}|_{\frac{r}{2}}(\gamma_1\gamma_2) \Rightarrow$

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Beweis.

$\Theta_{A,\lambda}(T.\tau) = \Theta_{A,\lambda}(\tau + 1) \in \mathrm{span}_{\mathbb{C}}(V) \wedge$
$\Theta_{A,\lambda}(S.\tau) = \Theta_{A,\lambda}\left(-\frac{1}{\tau}\right) \in \mathrm{span}_{\mathbb{C}}(V) \wedge \Gamma_1 = \langle T, S \rangle \wedge$
$|$ linkslinear, $\forall \gamma_1, \gamma_2 \in \Gamma_1 : (\Theta_{A,\lambda}|_{\frac{r}{2}}(\gamma_1))|_{\frac{r}{2}}(\gamma_2) = \Theta_{A,\lambda}|_{\frac{r}{2}}(\gamma_1\gamma_2) \Rightarrow$

$$(\Theta_{A,\lambda}|_{\frac{r}{2}}(\gamma))(\tau) = \frac{1}{(c\tau + d)^{\frac{r}{2}}} \Theta_{A,\lambda}(\gamma.\tau) \in \mathrm{span}_{\mathbb{C}}(V)$$

□

Hilfsdefinition

Definition

$A \in M_{r,r}(\mathbb{Z})$ gerade; $\lambda \in A^{-1}\mathbb{Z}^r$; $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$; $c \neq 0 \to$

$$S_\gamma(\lambda) := \sum_{\mu \in \mathbb{Z}^r / c\mathbb{Z}^r} \exp\left(2\pi i \frac{a}{c} Q(\mu + \lambda)\right)$$

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$\Rightarrow S_\gamma(\lambda)$ wohldefiniert (unabhängig von Restklassenwahl)$\wedge$
$\forall \lambda \in A^{-1}\mathbb{Z}^r / \mathbb{Z}^r : S_\gamma(\lambda + \mathbb{Z}^r) = S_\gamma(\lambda)$

Achtung

Strichformel

Theorem (Strichformel)

$r \in \mathbb{N}$ gerade; $A \in M_{r,r}(\mathbb{Z})$ gerade, positiv definit; $\lambda \in A^{-1}\mathbb{Z}^r$;
$\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$; $c \neq 0 \Rightarrow$

$$\Theta_{A,\lambda}|_{\frac{r}{2}}(\gamma) = \frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{2}}\sqrt{\det(A)}} \cdot \sum_{\lambda' \in A^{-1}\mathbb{Z}^r/\mathbb{Z}^r} \exp(-2\pi i[bB(\lambda,\lambda') + bdQ(\lambda')]) \cdot S_\gamma(\lambda + d\lambda') \cdot \Theta_{A,\lambda'}$$

1. $S_{-\gamma} = S_{\gamma} \wedge |_{\frac{r}{2}}(-\gamma) = (-1)^{\frac{r}{2}}|_{\frac{r}{2}}(\gamma) \Rightarrow \gamma - > -\gamma \Rightarrow$ Œ $\quad c > 0$

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$$\exp(\ldots)\frac{(-i)^{\frac{r}{2}}}{\sqrt{\det(cA)}} \sum_{\nu \in (cA)^{-1}\mathbb{Z}^r/\mathbb{Z}^r} \exp(2\pi i B(\lambda+\mu,\nu))\Theta_{cA,\nu}$$

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10. $8 \wedge 9 \wedge \mu \mapsto \mu - d\mu' \Rightarrow$

9. $x := \lambda + \mu, y := \lambda' + \mu' \wedge$
$\frac{a}{c}Q(x)+\frac{1}{c}B(x,y)+\frac{d}{c}Q(y) = \frac{a}{c}Q(x+dy)-bB(x,y)-bdQ(y)\wedge$
$B(\lambda+\mu,\lambda'+\mu')-B(\lambda,\lambda') = B(\lambda,\lambda')+B(\lambda',\mu)+B(\mu,\mu') \in \mathbb{Z}\wedge$
$Q(\lambda'+\mu') - Q(\lambda') = B(\lambda',\mu') + Q(\mu') \in \mathbb{Z} \Rightarrow$

$$\exp\left(2\pi i\left[\frac{a}{c}Q(\lambda+\mu)+\frac{1}{c}B(\lambda+\mu,\lambda'+\mu')+\frac{d}{c}Q(\lambda'+\mu')\right]\right) =$$
$$\exp\left(2\pi i\left[\frac{a}{c}Q(\lambda+\mu+d\lambda'+d\mu')-bB(\lambda,\lambda')-bdQ(\lambda')\right]\right)$$

10. $8 \wedge 9 \wedge \mu \mapsto \mu - d\mu' \Rightarrow$

$$\Theta_{cA,\lambda}|_{\frac{r}{2}}\left(V_c^{-1}T^aST^d\right)$$

9. $x := \lambda + \mu, y := \lambda' + \mu' \wedge$
$\frac{a}{c}Q(x) + \frac{1}{c}B(x,y) + \frac{d}{c}Q(y) = \frac{a}{c}Q(x+dy) - bB(x,y) - bdQ(y) \wedge$
$B(\lambda+\mu, \lambda'+\mu') - B(\lambda,\lambda') = B(\lambda,\lambda') + B(\lambda',\mu) + B(\mu,\mu') \in \mathbb{Z} \wedge$
$Q(\lambda'+\mu') - Q(\lambda') = B(\lambda',\mu') + Q(\mu') \in \mathbb{Z} \Rightarrow$

$$\exp\left(2\pi i\left[\frac{a}{c}Q(\lambda+\mu) + \frac{1}{c}B(\lambda+\mu, \lambda'+\mu') + \frac{d}{c}Q(\lambda'+\mu')\right]\right) =$$
$$\exp\left(2\pi i\left[\frac{a}{c}Q(\lambda+\mu+d\lambda'+d\mu') - bB(\lambda,\lambda') - bdQ(\lambda')\right]\right)$$

10. $8 \wedge 9 \wedge \mu \mapsto \mu - d\mu' \Rightarrow$

$$\Theta_{cA,\lambda}|_{\frac{r}{2}}\left(V_c^{-1}T^aST^d\right) = \frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{4}}\sqrt{\det(cA)}} \cdot \sum_{\substack{\mu,\mu' \in \mathbb{Z}_c^r \\ \lambda' \in A^{-1}\mathbb{Z}^r/\mathbb{Z}^r}}$$
$$\exp\left(2\pi i\left[\frac{a}{c}Q(\lambda+\mu+d\lambda') - bB(\lambda,\lambda') - bdQ(\lambda')\right]\right)\Theta_{cA,\frac{\lambda'+\mu'}{c}}$$

11.

$$\Theta_{cA,\lambda}|_{\frac{r}{2}}\left(V_c^{-1}T^aST^d\right)$$

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$$\Theta_{cA,\lambda}|_{\frac{r}{2}}\left(V_c^{-1}T^aST^d\right)$$

$$\underline{\underline{10}}\ \frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{4}}\sqrt{\det(cA)}}\frac{c^{-\frac{r}{4}}}{c^{-\frac{r}{4}}}\sum_{\substack{\mu\in\mathbb{Z}_c^r\\ \lambda'\in A^{-1}\mathbb{Z}^r/\mathbb{Z}^r}}$$

$$\exp\left(2\pi i\left[\frac{a}{c}Q(\lambda+\mu+d\lambda')-bB(\lambda,\lambda')-bdQ(\lambda')\right]\right)\sum_{\mu'\in\mathbb{Z}_c^r}\Theta_{cA,\frac{\lambda'+\mu'}{c}}$$

11.

$$\Theta_{cA,\lambda}\big|_{\frac{r}{2}}\left(V_c^{-1}T^aST^d\right)$$

$$\overset{10}{=} \frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{4}}\sqrt{\det(cA)}}\frac{c^{-\frac{r}{4}}}{c^{-\frac{r}{4}}} \sum_{\substack{\mu\in\mathbb{Z}_c^r\\ \lambda'\in A^{-1}\mathbb{Z}^r/\mathbb{Z}^r}}$$

$$\exp\left(2\pi i\left[\frac{a}{c}Q(\lambda+\mu+d\lambda')-bB(\lambda,\lambda')-bdQ(\lambda')\right]\right)\sum_{\mu'\in\mathbb{Z}_c^r}\Theta_{cA,\frac{\lambda'+\mu'}{c}}$$

$$\overset{3}{=} \frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{2}}\sqrt{\det(A)}} \sum_{\substack{\mu\in\mathbb{Z}_c^r\\ \lambda'\in A^{-1}\mathbb{Z}^r/\mathbb{Z}^r}}$$

$$\exp\left(2\pi i\left[\frac{a}{c}Q(\lambda+\mu+d\lambda')-bB(\lambda,\lambda')-bdQ(\lambda')\right]\right)\Theta_{A,\lambda'}\big|_{\frac{r}{2}}\left(V_c^{-1}\right)$$

12.

$$\Theta_{cA,\lambda}|_{\frac{r}{2}}\left(V_c^{-1}T^aST^d\right)$$

12.

$$\Theta_{cA,\lambda}|_{\frac{r}{2}}\left(V_c^{-1}T^aST^d\right)$$
$$\stackrel{11}{=} \frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{2}}\sqrt{\det(A)}} \sum_{\lambda'\in A^{-1}\mathbb{Z}^r/\mathbb{Z}^r} \exp(-2\pi i[bB(\lambda,\lambda')+bdQ(\lambda')])\cdot$$
$$\left[\sum_{\mu\in\mathbb{Z}_c^r}\exp\left(\frac{a}{c}Q(\mu+\lambda+d\lambda')\right)\right]\Theta_{A,\lambda'}|_{\frac{r}{2}}\left(V_c^{-1}\right)$$

12.

$$\Theta_{cA,\lambda}|_{\frac{r}{2}}\left(V_c^{-1}T^aST^d\right)$$

$$\overset{11}{=}\frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{2}}\sqrt{\det(A)}}\sum_{\lambda'\in A^{-1}\mathbb{Z}^r/\mathbb{Z}^r}\exp(-2\pi i[bB(\lambda,\lambda')+bdQ(\lambda')])\cdot$$

$$\left[\sum_{\mu\in\mathbb{Z}_c^r}\exp\left(\frac{a}{c}Q(\mu+\lambda+d\lambda')\right)\right]\Theta_{A,\lambda'}|_{\frac{r}{2}}\left(V_c^{-1}\right)$$

$$=\frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{2}}\sqrt{\det(A)}}\sum_{\lambda'\in A^{-1}\mathbb{Z}^r/\mathbb{Z}^r}\exp(-2\pi i[bB(\lambda,\lambda')+bdQ(\lambda')])\cdot$$

$$S_\gamma(\lambda+d\lambda')\Theta_{A,\lambda'}|_{\frac{r}{2}}\left(V_c^{-1}\right)$$

12.

$$\Theta_{cA,\lambda}|_{\frac{r}{2}} \left(V_c^{-1} T^a S T^d\right)$$

$$\stackrel{11}{=} \frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{2}}\sqrt{\det(A)}} \sum_{\lambda' \in A^{-1}\mathbb{Z}^r/\mathbb{Z}^r} \exp(-2\pi i[bB(\lambda,\lambda') + bdQ(\lambda')])\cdot$$

$$\left[\sum_{\mu \in \mathbb{Z}_c^r} \exp\left(\frac{a}{c} Q(\mu + \lambda + d\lambda')\right)\right] \Theta_{A,\lambda'}|_{\frac{r}{2}} \left(V_c^{-1}\right)$$

$$= \frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{2}}\sqrt{\det(A)}} \sum_{\lambda' \in A^{-1}\mathbb{Z}^r/\mathbb{Z}^r} \exp(-2\pi i[bB(\lambda,\lambda') + bdQ(\lambda')])\cdot$$

$$S_\gamma(\lambda + d\lambda')\Theta_{A,\lambda'}|_{\frac{r}{2}} \left(V_c^{-1}\right)$$

13. $2, 11 \Rightarrow \Theta_{A,\lambda}|_{\frac{r}{2}}(\gamma) = \Theta_{cA,\frac{\lambda+\mu}{c}}|_{\frac{r}{2}} \left(V_c^{-1} T^a S T^d V_c\right)$ □

Ende

Vielen Dank für Ihre Aufmerksamkeit!