Approaching NSP-EMD with B-Splines

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introduction

intrinsic mode function (IMF) (1/2)

definition

Let $a, \phi \in \mathcal{C}_b^2(\mathbb{R})$ with bounded derivatives.

$$a(t)cos(\phi(t)) :\in \mathcal{I}$$

is an IMF with accuracy $\varepsilon \in (0,1)$ if and only if for all $t \in \mathbb{R}$

- $a(t) > 0$, $\phi'(t) > 0$,
- $|a'(t)| \leq \varepsilon|\phi'(t)|$,
- $|\phi''(t)| \leq \varepsilon|\phi'(t)|$.

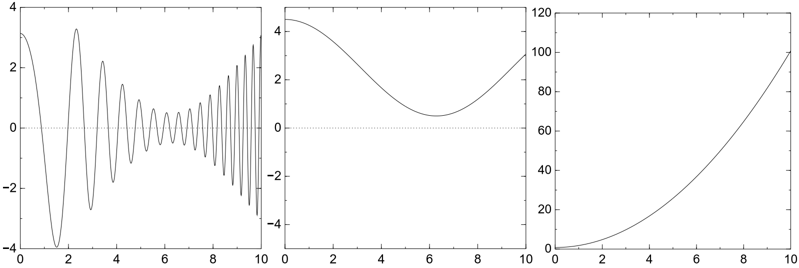
heuristic

- Physical reasonability.
- Slowly varying amplitude.
- Slowly varying frequency.

introduction

intrinsic mode function (IMF) (2/2)

example



$a(t)cos(\phi(t))$ $a(t)$ $\phi(t)$

introduction

empirical mode decomposition (EMD) (N.E. Huang et al., 1998)

input

$$\text{signal } s\colon [p,q] \to \mathbb{R}$$

decomposition

$$s(t) = \sum_{k=0}^{w} s_k(t) + r_{w+1}(t), \quad s_k \in \mathcal{I}$$
$$s_k(t) = a_k(t) cos(\phi_k(t))$$

iterative

$$r_0(t) = s(t)$$
$$r_{k+1}(t) = r_k(t) - s_k(t)$$

goal

Find a_k and ϕ_k.

introduction

operator-based signal separation (OSS) (S. Peng et al., 2008) (1/2)

idea

Let $P_k = P_k(a_k, \phi_k)$, $Q_k = Q_k(a_k, \phi_k)$, $R_k = R_k(a_k, \phi_k)$.
Find differential operator $\mathcal{D}_{P_k,Q_k,R_k}$ analytically such that

$$\mathcal{D}_{P_k,Q_k,R_k} s_k = 0.$$

Solve

$$\begin{aligned}(P_k, Q_k, R_k) = \arg \min_{(\tilde{P},\tilde{Q},\tilde{R})} \quad & \|\mathcal{Q}_1(r_k - \tilde{s})\| \\ \text{s.t.} \quad & \mathcal{D}_{\tilde{P},\tilde{Q},\tilde{R}} \tilde{s} = 0 \\ & \|\mathcal{Q}_2 \tilde{P}\| \leq \tau \\ & \|\mathcal{Q}_3 \tilde{Q}\| \leq \tau \\ & \|\mathcal{Q}_4 \tilde{R}\| \leq \tau \end{aligned}$$

with $\tau > 0$, $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4 \in \{D^0, D^1, \dots\}$ regularization operators, $\tilde{P} := P_k(\tilde{a}, \tilde{\phi})$, $\tilde{Q} := Q_k(\tilde{a}, \tilde{\phi})$, $\tilde{R} := R_k(\tilde{a}, \tilde{\phi})$ and $\tilde{s} := \tilde{a}(t) cos(\tilde{\phi}(t))$.

introduction

operator-based signal separation (OSS) (S. Peng et al., 2008) (2/2)

unconstrained formulation

$$
\begin{aligned}
(P_k, Q_k, R_k) = \arg \min_{(\tilde{P},\tilde{Q},\tilde{R})} \|\mathcal{D}_{\tilde{P},\tilde{Q},\tilde{R}}\tilde{s}\| + \\
\lambda \|\mathcal{Q}_1(r_k - \tilde{s})\| + \\
\mu(\|\mathcal{Q}_2\tilde{P}\| + \|\mathcal{Q}_3\tilde{Q}\| + \|\mathcal{Q}_4\tilde{R}\|)
\end{aligned}
$$

problem

Want to control $\tilde{s}$, and thus s_k, in some way.

introduction

null space pursuit (NSP) (S. Peng, W.-L. Hwang et al., 2010)

solution

Introduce leakage factor γ.

$$(P_k, Q_k, R_k) = \arg \min_{(\tilde{P},\tilde{Q},\tilde{R})} \|\mathcal{D}_{\tilde{P},\tilde{Q},\tilde{R}}\tilde{s}\| + \lambda(\|\mathcal{Q}_1(r_k - \tilde{s})\| + \gamma\|\tilde{s}\|) + \mu(\|\mathcal{Q}_2\tilde{P}\| + \|\mathcal{Q}_3\tilde{Q}\| + \|\mathcal{Q}_4\tilde{R}\|)$$

Controls retainment of information in residual signal.

remaining problems (OSS, NSP)

- Processing necessary for $(\tilde{P}, \tilde{Q}, \tilde{R}) \to \tilde{s}$.
- Very heuristical and thus hard to analyze.
- Does not explicitly enforce IMF-properties.

Does it really help understanding EMD?

vanishing operators

example (X. Hu, S. Peng, W.-L. Hwang, 2012)

$$\begin{aligned} P_k(a_k, \phi_k) &= -2\frac{a_k'}{a_k}, \\ Q_k(a_k, \phi_k) &= (\phi')^2 + 2\frac{a_k'}{a_k}, \\ R_k(a_k, \phi_k) &= 0, \\ \mathcal{D}_{P_k,Q_k,R_k} &= \frac{d^2}{dt^2} + P_k(t)\frac{d}{dt} + Q_k(t). \end{aligned}$$

problem

Let $a_k''(t) \neq 0$. It holds

$$\mathcal{D}_{P_k,Q_k,R_k} s_k(t) = \frac{a_k''(t)}{a_k(t)} \neq 0.$$

vanishing operators

example (Guo et al., 2017)

solution

Let $A_k(a_k) := \frac{a_k'}{a_k}$, $\Omega_k(\phi_k) := \frac{1}{(\phi_k')^2}$.

$$P_k(A_k, \Omega_k) = \Omega_k$$

$$Q_k(A_k, \Omega_k) = -2P_k A_k + \frac{1}{2}\frac{d}{dt}P_k$$

$$R_k(A_k, \Omega_k) = P_k\left(A_k^2 - \frac{d}{dt}A_k\right) - \frac{1}{2}\frac{d}{dt}P_k A_k + 1$$

$$\mathcal{D}_{P_k,Q_k,R_k} = P_k(t)\frac{d^2}{dt^2} + Q_k(t)\frac{d}{dt} + R_k(t).$$

It holds $\mathcal{D}_{P_k,Q_k,R_k} s_k(t) = 0$.

problem

Yet another complicating step $(A_k, \Omega_k) \to (a_k, \phi_k)$.

Can we find a simple general form?

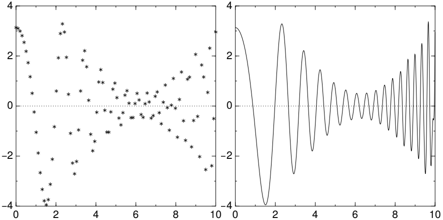
B-Splines

approach

- Express functions, e.g. the signal, as B-splines of order $k \geq 1$:

$$s(t) = \sum_{i=0}^{n-1} s_i B_{i,k}(t) \in \mathcal{C}^{k-2}.$$

- Least-Squares-Fit of time series data.



- Generate extended grid with $q \geq 0$ intermediate points between spline knots.
- Preprocess $B_{i,k}$ and its derivatives (e.g. with GLS's gsl_bspline_deriv_eval_nonzero()) on extended grid.
- Directly optimize over (a_k, ϕ_k).

B-Splines

advantages

- Sparse Least-Square-fit- and evaluation-matrices (even of derivatives) (local support of $B_{i,k}$ and its derivatives).
- B-Splines are already very smooth (do we even need $\|\mathcal{Q}_2\tilde{P}\| \le \tau$, $\|\mathcal{Q}_3\tilde{Q}\| \le \tau$, $\|\mathcal{Q}_4\tilde{R}\| \le \tau$ any more?).
- Derivatives are free. Let $m \le k-2$:

$$s^{(m)}(t) = \sum_{i=0}^{n-1} s_i B_{i,k}^{(m)}(t).$$

- Constant time evaluation of $s_k^{(m)}$ only with a_k and ϕ_k using LEIBNIZ' rule and FAÀ DI BRUNO's formula after preprocessing multinomials and partitions.

B-Splines

NSP reinvestigation

Let $\tilde{s}(t) := \tilde{a}(t)cos(\tilde{\phi}(t))$. The NSP reformulated with B-Splines is

$$(a_k, \phi_k) = \arg\min_{(\tilde{a},\tilde{\phi})} \|\mathcal{D}_{\tilde{a},\tilde{\phi}}\tilde{s}\| + \lambda(\|\mathcal{Q}_1(r_k - \tilde{s})\| + \gamma\|\tilde{s}\|)$$

Do we even need the differential operator?

- Operator adapted to get smooth $\tilde{P}$, $\tilde{Q}$ and $\tilde{R}$.
- $\tilde{a}$ and $\tilde{\phi}$ are already smooth by definition.

reduced system

$$(a_k, \phi_k) = \arg\min_{(\tilde{a},\tilde{\phi})} \|\mathcal{Q}_1(r_k - \tilde{s})\| + \gamma\|\tilde{s}\|$$

We can do better than that.

proposed optimization problem

IMF requirements for $t \in \mathbb{R}$

- a_k, ϕ_k bounded and with bounded derivatives (given)
- $a_k(t) > 0$, $\phi'_k(t) > 0$
- $|a'_k(t)| \leq \varepsilon|\phi'_k(t)|$, $|\phi''_k(t)| \leq \varepsilon|\phi'_k(t)|$

Let $c(\tilde{a}, \tilde{\phi}) \in \mathbb{R}$ be a cost function and $\varepsilon > 0$ fixed.

$$\begin{aligned}
(a_k, \phi_k) = \arg\min_{(\tilde{a},\tilde{\phi})} \quad & c(\tilde{a}, \tilde{\phi}) \\
\text{s.t.} \quad & \tilde{a}(t) > 0 \\
& \tilde{\phi}'(t) > 0 \\
& |\tilde{a}'(t)| \leq \varepsilon|\tilde{\phi}'(t)| \\
& |\tilde{\phi}''(t)| \leq \varepsilon|\tilde{\phi}'(t)|
\end{aligned}$$

We can e.g. set $c(\tilde{a}, \tilde{\phi}) = \|\mathcal{Q}(r_k - \tilde{s})\|$ with $Q \in \{D^0, \ldots, D^{k-2}\}$.

conclusion

- Differential operator might not be necessary with B-Splines.
- Proposed problem has strict separation of fitness and qualification.
- Great flexibility of cost-functions due to power of preprocessing.

outlook

- Implement optimization problem.
- Investigate other cost functions.
- Compare with other approaches.