Discourse

Generating Simple Givens Rotation Examples with Odd Perfect Squares

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1 Introduction

1.1 Givens rotation

The basic idea behind the Givens roation is to eliminate an entry in a vector. This is done by rotating it in two dimensions, effectively modifying the vector only at two entries. Without loss of generality we can thus just look at the reduced problem of rotating a two-dimensional vector $(a, b)^T$ into a scaled unit vector $(r, 0)^T$, effectively eliminating the entry b. Given this is meant to be a rotation, we want to preserve the norm and thus enforce $r = \sqrt{a^2 + b^2}$.

The rotation matrix entries are denoted as c and s, yielding the general algebraic formulation

$$\begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} r \\ 0 \end{pmatrix}.$$

It can then be derived that the entries c and s have to be chosen as

$$c = \frac{a}{\sqrt{a^2 + b^2}}$$

and

$$s = \frac{b}{\sqrt{a^2 + b^2}}.$$

For optimal numerical stability though these are not naïvely calculated but with Algorithm 1.1 which we will also as follows assume to be used for calculating the Givens rotation.

Algorithm 1.1 (Stable Givens parameter calculation)**.**

if $b = 0$ ***then***
  $c \leftarrow 1$
  $s \leftarrow 0$
else if $|b| > |a|$ ***then***
  $\tau \leftarrow \frac{a}{b}$
  $s \leftarrow \frac{1}{\sqrt{1+\tau^2}}$
  $c \leftarrow s\tau$
else
  $\tau \leftarrow \frac{b}{a}$
  $c \leftarrow \frac{1}{\sqrt{1+\tau^2}}$
  $s \leftarrow c\tau$
end if

1.2 Requirements

It is important to be able to reliably generate examples for algorithms like this such that in the context of an examination there is a high variance of tasks while still allowing all intermediate results to be calculated without the help of a calculator and in analytical form.

In the context of this discourse a simple example is meant to be one involving few Givens-rotations to bring into upper triagonal form and where the intermediate variables and results in Algorithm 1.1, the chosen matrix entries in the example itself and the final results all have a simple numerical form, i.e. natural or rational numbers.

1.3 General Formulation

It is valid to assume that at most two iterations are sufficient for an example to be understood. The minimal vertical dimension is 3, while it is also not necessary to give a square matrix, as it would involve rotating 3 vectors in each iteration. All in all, there is only one canonical form for an example that satisfied the requirements:

$$\begin{pmatrix} \star & \star \\ 0 & \star \\ 0 & 0 \\ \star & \star \end{pmatrix} \tag{1.1}$$

It has the desired problem size and because the two entries at the bottom that are to be eliminated are in the same line, both iterations are related such that a mistake in the first iteration fundamentally affects the result in the second iteration. To make it easier to grasp we give names to the parameters of the example matrix in Equation (1.1) without yet giving an indication on their domain.

$$\begin{pmatrix} x & y \\ 0 & z \\ 0 & 0 \\ m & n \end{pmatrix} \tag{1.2}$$

Definition 1.2 (Perfect Square)**.** *Let $n \in \mathbb{N}$. n is a* perfect square *if and only if*

$$\exists_{m\in\mathbb{N}} : n = m^2.$$

Proposition 1.3 (Odd Number Theorem)**.** *Let $n \in \mathbb{N}$ be a perfect square. It holds that*

$$n = \sum_{i=1}^{\sqrt{n}} (2i - 1).$$

Proof. By induction. □

2 Third Case Generation Procedure

In this Subsection, we generate the matrix such that $|m| < |x|$ and we thus hit the third case on the frist rotation in the generation scheme of Algorithm 1.1.

2.1 First Rotation

To begin with the generation, choose an arbitrary odd number $\tilde{t}_1 > 1$, yielding the odd perfect square $\tilde{q}_1 := \tilde{t}_1^2 > 1$ and set

$$p_1 := \tilde{q}_1 - 2.$$

As p_1 is also odd by construction there exists an even number $t_1 \in \mathbb{N}$ such that

$$p_1 = 2t_1 - 1.$$

As a side note, we know that

$$\tilde{q}_1 - 2 = \tilde{t}_1^2 - 2 = p_1 = 2t_1 - 1,$$

which is equivalent to

$$\boxed{t_1 = \frac{\tilde{t}_1^2 - 1}{2}} \in \mathbb{N}. \tag{2.1}$$

We generate another perfect square (by Proposition 1.3)

$$q_1 := \sum_{\substack{i=1 \\ i \text{ odd}}}^{p_1} i = \sum_{i=1}^{t_1} (2i - 1) = t_1^2$$

and set the entries of the first column to be

$$\boxed{x := \sqrt{q_1} = t_1} \in \mathbb{N}$$

and

$$\boxed{m := \sqrt{p_1 + 2} = \sqrt{\tilde{q}_1} = \tilde{t}_1} \in \mathbb{N}.$$

Applying Algorithm 1.1 with $a = x$ and $b = m$ we first observe that $|m| < |x|$ by construction and thus note that we enter the third 'else'-case. It follows by construction that

$$\boxed{\tau_1 = \frac{b}{a} = \frac{m}{x} = \frac{\tilde{t}_1}{t_1}} \in \mathbb{Q}$$

and

$$\begin{aligned}
c_1 &= \frac{1}{\sqrt{1 + \tau^2}} \\
&= \frac{1}{\sqrt{1 + \frac{m^2}{x^2}}} \\
&= \frac{1}{\sqrt{1 + \frac{p_1+2}{q_1}}} \\
&= \frac{1}{\sqrt{\frac{q_1}{q_1} + \frac{p_1+2}{q_1}}} \\
&= \frac{1}{\sqrt{\frac{q_1+p_1+2}{q_1}}} \\
&= \frac{\sqrt{q_1}}{\sqrt{q_1 + (p_1 + 2)}}.
\end{aligned}$$

By construction (q_1 is the sum of all odd integers up to p_1) and Propositon 1.3 it follows that $q_1 + (p_1 + 2)$ is a perfect square as well, more precisely

$$q_1 + (p_1 + 2) = (t_1 + 1)^2,$$

and we thus obtain

$$\boxed{c_1 = \frac{t_1}{t_1 + 1}} \in \mathbb{Q}.$$

The second GIVENS parameter s_1 follows immediately as

$$\boxed{s_1 = c_1 \tau_1 = \frac{t_1 \tilde{t}_1}{(t_1 + 1) t_1} = \frac{\tilde{t}_1}{t_1 + 1}} \in \mathbb{Q}$$

For the resulting vector, we obtain the parameter r_1 as

$$\boxed{r_1 = \sqrt{x^2 + m^2} = \sqrt{q_1 + (p_1 + 2)} = t_1 + 1} \in \mathbb{N}$$

2.2 Applying the First Rotation to the Second Column

Putting aside the choice of y and n for later, we apply the first rotation to these affected entries (as they are in the same row as x and m).

$$\begin{pmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{pmatrix} \begin{pmatrix} y \\ n \end{pmatrix} = \begin{pmatrix} \tilde{y} \\ \tilde{n} \end{pmatrix}$$

We do not really care about the final form of $\tilde{y}$ but note here that if we at least choose y and n from $\mathbb{Q}$ we are assured that $\tilde{y} \in \mathbb{Q}$. To keep it simple, we want to proceed in this second column analogously to the first column, which means that we want to do with z and $\tilde{n}$ what we did with x and m in the first column. To achieve this goal, we need $\tilde{n}$ to be odd and strictly larger than 1 so $\tilde{n}^2$ is an odd perfect square.

$$\tilde{n} = -s_1 y + c_1 n = -\frac{\tilde{t}_1}{t_1 + 1} y + \frac{t_1}{t_1 + 1} n = \frac{1}{t_1 + 1} \left(t_1 n - \tilde{t}_1 y \right)$$

To make $\tilde{n}$ an odd number that is strictly larger than 3, we have to turn the numerator into an odd multiple of $t_1 + 1$. This can be done by setting for an odd natural $\alpha \geq 1$

$$\boxed{y = -\alpha} \in \mathbb{N}$$

and

$$\boxed{n = \alpha \tilde{t}_1} \in \mathbb{N},$$

and as $\tilde{t}_1 > 1$ and α are odd we obtain

$$\boxed{\tilde{n} = \frac{\alpha \tilde{t}_1 (t_1 + 1)}{t_1 + 1} = \alpha \tilde{t}_1} \in \mathbb{N}$$

as an odd number strictly larger than 1. We can now also calculate $\tilde{y}$ with Equation (2.1) as

$$\boxed{\tilde{y} = c_1 y + s_1 n = \frac{-\alpha t_1}{t_1 + 1} + \frac{\tilde{t}_1 \alpha \tilde{t}_1}{t_1 + 1} = \frac{\alpha(\tilde{t}_1^2 - t_1)}{t_1 + 1} = \frac{\alpha \left(\frac{\tilde{t}_1^2}{2} + \frac{1}{2} \right)}{\frac{\tilde{t}_1^2}{2} + \frac{1}{2}} = \alpha} \in \mathbb{N}.$$

2.3 Second Rotation

The parameter z is unaffected from the first rotation and we can choose it as we like, which we'll use to rotate $(z, \tilde{n})^T$ to $(r_2, 0)^T$. Analogous to the first rotation we interpret our $\tilde{t}_2 := \tilde{n}$ to be the square root of an odd perfect square $\tilde{q}_2 > 1$, set

$$p_2 := \tilde{q}_2 - 2$$

and again find an even number $t_2 \in \mathbb{N}$ such that

$$p_2 = 2t_2 - 1.$$

As a side note, we know that

$$\tilde{q}_2 - 2 = \tilde{t}_2^2 - 2 = p_2 = 2t_2 - 1,$$

which is equivalent to

$$\boxed{t_2 = \frac{\tilde{t}_2^2 - 1}{2} = \frac{\alpha^2 \tilde{t}_1^2 - 1}{2}} \in \mathbb{N}.$$

We generate another perfect square (by Proposition 1.3)

$$q_1 := t_2^2$$

and set the entry z of the second column to be

$$\boxed{z := \sqrt{q_2} = t_2}.$$

Applying Algorithm 1.1 with $a = z$ and $b = \tilde{n}$ we first observe that $|\tilde{n}| < |z|$ by construction and thus note that we enter the third 'else'-case just like with the first rotation. We yield analogously

$$\boxed{\tau_2 = \frac{b}{a} = \frac{\tilde{n}}{z} = \frac{\tilde{t}_2}{t_2} = \frac{\alpha \tilde{t}_1}{t_2}} \in \mathbb{Q},$$

$$\boxed{c_2 = \frac{t_2}{t_2 + 1}} \in \mathbb{Q},$$

$$\boxed{s_2 = c_2 \tau_2 = \frac{\tilde{t}_2}{t_2 + 1} = \frac{\alpha \tilde{t}_1}{t_2 + 1}} \in \mathbb{Q},$$

and

$$\boxed{r_2 = \sqrt{z^2 + \tilde{n}^2} = \sqrt{q_2 + (p_2 + 2)} = t_2 + 1} \in \mathbb{Z}.$$

2.4 Examples

For $\tilde{t}_1 = 3$ and $\alpha = 1$ we obtain $t_1 = 4$, $t_2 = 4$ and the matrix

$$\begin{pmatrix} 4 & -1 \\ 0 & 4 \\ 0 & 0 \\ 3 & 3 \end{pmatrix} \to \begin{pmatrix} 5 & 1 \\ 0 & 4 \\ 0 & 0 \\ 0 & 3 \end{pmatrix} \to \begin{pmatrix} 5 & 1 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

with the intermediate results $\tau_1 = \frac{3}{4}$, $c_1 = \frac{4}{5}$, $s_1 = \frac{3}{5}$, $\tau_2 = \frac{3}{4}$, $c_2 = \frac{4}{5}$ and $s_2 = \frac{3}{5}$. We notice that the intermediate results also all have a nice form and thus the GIVENS rotations for this matrix are easy to calculate by hand.

For $\tilde{t}_1 = 3$ and $\alpha = 3$ we obtain $t_1 = 4$, $t_2 = 40$ and the matrix

$$\begin{pmatrix} 4 & -3 \\ 0 & 40 \\ 0 & 0 \\ 3 & 9 \end{pmatrix} \to \begin{pmatrix} 5 & 3 \\ 0 & 40 \\ 0 & 0 \\ 0 & 9 \end{pmatrix} \to \begin{pmatrix} 5 & 3 \\ 0 & 41 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

For $\tilde{t}_1 = 5$ and $\alpha = 1$ we obtain $t_1 = 12$, $t_2 = 12$ and the matrix

$$\begin{pmatrix} 12 & -1 \\ 0 & 12 \\ 0 & 0 \\ 5 & 5 \end{pmatrix} \to \begin{pmatrix} 13 & 1 \\ 0 & 12 \\ 0 & 0 \\ 0 & 5 \end{pmatrix} \to \begin{pmatrix} 13 & 1 \\ 0 & 13 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

3 Outlook

One can variate the deduction by swapping the entries of the column so the second case of Algorithm 1.1 is triggered. However, this will only minimally influence the end result.

Another variation is the approach to calculating $\tilde{n}$ when applying the second rotation. This aspect is where the most variation can be found, and it is entirely possible to parametrize n and y differently (e.g. both as multiples of $t_1 + 1$).

Diving a bit deeper, one could also not just look at adding the single next odd number to the perfect square, but pairs or arbitrary sums of odd numbers directly to generate another perfect square and increase the variation of the possible examples even more.