

Thetafunktionen positiv definiter quadratischer Formen III

Thetafunktionen mit Charakteristik

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Wiederholung

Definition (Gerade Matrix)

$r \in \mathbb{N}$; $A \in M_{r,r}(\mathbb{Z})$ symmetrisch \rightarrow

A gerade $:\Leftrightarrow \forall i \in \{1, \dots, r\} : A_{ii}$ gerade

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Definition (PETERSSONScher Strichoperator)

$k \in \mathbb{Z}$; $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2,2}(\mathbb{R})$; $f : \mathbb{H} \rightarrow \mathbb{C}$ holomorph \rightarrow

$$(f|_k(\gamma))(\tau) := \frac{\det(\gamma)^{\frac{k}{2}}}{(c\tau + d)^k} f(\gamma.\tau) \quad \text{mit} \quad \gamma.\tau := \frac{a\tau + b}{c\tau + d}$$

Definition (A -assoziierte Formen)

$r \in \mathbb{N}$; $A \in M_{r,r}(\mathbb{Z})$ gerade \rightarrow

Assoziierte quadratische Form: $Q : \mathbb{R}^r \rightarrow \mathbb{R}$, $Q(x) := \frac{1}{2}x^T Ax$

Assoziierte Bilinearform: $Q : \mathbb{R}^r \times \mathbb{R}^r \rightarrow \mathbb{R}$, $B(x, y) := x^T Ay$

$$Q_A^* := Q^*(x) := Q_{A^{-1}}(x)$$

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Definition (A -assoziierte Thetafunktion)

$r \in \mathbb{N}$; $A \in M_{r,r}(\mathbb{Z})$ gerade, positiv definit \rightarrow

$$\forall \tau \in \mathbb{H} : q := \exp(2\pi i \tau) : \Theta_A(\tau) := \sum_{n \in \mathbb{Z}^r} q^{Q(n)}$$

Definition

Definition (A -assoziierte Thetafunktion mit Charakteristik λ)

$r \in \mathbb{N}$; $A \in M_{r,r}(\mathbb{Z})$ gerade, positiv definit; $\lambda \in A^{-1}\mathbb{Z}^r \rightarrow$

$$\forall \tau \in \mathbb{H} : q := \exp(2\pi i\tau) : \Theta_{A,\lambda}(\tau) := \sum_{n \in \lambda + \mathbb{Z}^r} q^{Q(n)}$$

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Beweis.

$$\Theta_{A,0}(\tau) = \sum_{n \in 0 + \mathbb{Z}^r} q^{Q(n)} = \sum_{n \in \mathbb{Z}^r} q^{Q(n)} = \Theta_A(\tau) \quad \square$$

Bemerkung (Reduktion von λ)

$$\forall \mu \in \mathbb{Z}^r : \Theta_{A, \lambda + \mu} = \Theta_{A, \lambda}$$

→ Es reicht $\lambda \in A^{-1}\mathbb{Z}^r / \mathbb{Z}^r$ zu betrachten.

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Beweis.

$$\mu + \mathbb{Z}^r = \mathbb{Z}^r \Rightarrow$$

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Bemerkung (Lineare Abhängigkeit)

$$\forall \lambda \in A^{-1}\mathbb{Z}^r / \mathbb{Z}^r : \Theta_{A,-\lambda} = \Theta_{A,\lambda}$$

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Beweis.

$$\forall x \in \mathbb{R}^r : Q(-x) = Q(x) \wedge -\mathbb{Z}^r = \mathbb{Z}^r \Rightarrow$$

$$\begin{aligned} \Theta_{A,-\lambda}(\tau) &= \sum_{n \in -\lambda + \mathbb{Z}^r} q^{Q(n)} = \sum_{n \in -(-\lambda + \mathbb{Z}^r) = \lambda - \mathbb{Z}^r} q^{Q(-n)} \\ &= \sum_{n \in \lambda + \mathbb{Z}^r} q^{Q(-n)} = \sum_{n \in \lambda + \mathbb{Z}^r} q^{Q(n)} = \Theta_{A,\lambda}(\tau) \quad \square \end{aligned}$$

Transformationsformeln

Theorem (Transformationsformeln)

$r \in \mathbb{N}$; $A \in M_{r,r}(\mathbb{Z})$ gerade, positiv definit; $\lambda \in A^{-1}\mathbb{Z}^r \Rightarrow$

$$\Theta_{A,\lambda}(\tau + 1) = \exp(2\pi i Q(\lambda)) \Theta_{A,\lambda}(\tau) \quad (\text{i})$$

$$\Theta_{A,\lambda} \left(-\frac{1}{\tau} \right) = \frac{(-i\tau)^{\frac{r}{2}}}{\sqrt{\det(A)}} \sum_{\mu \in A^{-1}\mathbb{Z}^r / \mathbb{Z}^r} \exp(2\pi i B(\lambda, \mu)) \Theta_{A,\mu}(\tau) \quad (\text{ii})$$

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Lemma (1)

★ *Theorem* $\Rightarrow f(x) := \exp\left(-\frac{2\pi Q(x)}{t}\right) \rightarrow$

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Lemma (2)

$$f \in \mathcal{S}(\mathbb{R}^r) \Rightarrow \sum_{n \in \mathbb{Z}^r} f(n) = \sum_{n \in \mathbb{Z}^r} (\mathcal{F}f)(n)$$

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 &= \exp(2\pi i \lambda x) (\mathcal{F}f)(x) & \Big| & \quad \text{Lemma(1)} \\
 &= \frac{t^{\frac{r}{2}}}{\sqrt{\det(A)}} \exp(-2\pi t Q^*(x) + 2\pi i \lambda x)
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$r \in \mathbb{N}$ gerade; $A \in M_{r,r}(\mathbb{Z})$ gerade, positiv definit; $\lambda \in A^{-1}\mathbb{Z}^r \Rightarrow$

$$\forall \gamma \in \Gamma_1 : \Theta_{A,\lambda}|_{\frac{r}{2}}(\gamma) \in \text{span}_{\mathbb{C}} \left(V := \left\{ \Theta_{A,\mu} \mid \mu \in A^{-1}\mathbb{Z}^r / \mathbb{Z}^r \right\} \right)$$

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$$| \text{linkslinär, } \forall \gamma_1, \gamma_2 \in \Gamma_1 : (\Theta_{A,\lambda}|_{\frac{r}{2}}(\gamma_1))|_{\frac{r}{2}}(\gamma_2) = \Theta_{A,\lambda}|_{\frac{r}{2}}(\gamma_1\gamma_2) \Rightarrow$$

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Bemerkung (Erweiterung auf Γ_1)

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$$\forall \gamma \in \Gamma_1 : \Theta_{A,\lambda}|_{\frac{r}{2}}(\gamma) \in \text{span}_{\mathbb{C}} \left(V := \left\{ \Theta_{A,\mu} \mid \mu \in A^{-1}\mathbb{Z}^r / \mathbb{Z}^r \right\} \right)$$

Beweis.

$$\Theta_{A,\lambda}(T.\tau) = \Theta_{A,\lambda}(\tau + 1) \in \text{span}_{\mathbb{C}}(V) \wedge$$

$$\Theta_{A,\lambda}(S.\tau) = \Theta_{A,\lambda}\left(-\frac{1}{\tau}\right) \in \text{span}_{\mathbb{C}}(V) \wedge \Gamma_1 = \langle T, S \rangle \wedge$$

$$| \text{linkslin.}, \forall \gamma_1, \gamma_2 \in \Gamma_1 : (\Theta_{A,\lambda}|_{\frac{r}{2}}(\gamma_1))|_{\frac{r}{2}}(\gamma_2) = \Theta_{A,\lambda}|_{\frac{r}{2}}(\gamma_1\gamma_2) \Rightarrow$$

$$(\Theta_{A,\lambda}|_{\frac{r}{2}}(\gamma))(\tau) = \frac{1}{(c\tau + d)^{\frac{r}{2}}} \Theta_{A,\lambda}(\gamma.\tau) \in \text{span}_{\mathbb{C}}(V) \quad \square$$

Hilfsdefinition

Definition

$A \in M_{r,r}(\mathbb{Z})$ gerade; $\lambda \in A^{-1}\mathbb{Z}^r$; $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$; $c \neq 0 \rightarrow$

$$S_\gamma(\lambda) := \sum_{\mu \in \mathbb{Z}^r / c\mathbb{Z}^r} \exp\left(2\pi i \frac{a}{c} Q(\mu + \lambda)\right)$$

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$$\forall \mu \in \mathbb{Z}^r / c\mathbb{Z}^r : \exp\left(2\pi i \frac{a}{c} Q(\mu + c\mathbb{Z}^r)\right) = \exp\left(2\pi i \frac{a}{c} Q(\mu)\right)$$

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$\Rightarrow S_\gamma(\lambda)$ wohldefiniert (unabhängig von Restklassenwahl) \wedge

$$\forall \lambda \in A^{-1}\mathbb{Z}^r / \mathbb{Z}^r : S_\gamma(\lambda + \mathbb{Z}^r) = S_\gamma(\lambda)$$



Strichformel

Theorem (Strichformel)

$r \in \mathbb{N}$ gerade; $A \in M_{r,r}(\mathbb{Z})$ gerade, positiv definit; $\lambda \in A^{-1}\mathbb{Z}^r$;
 $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$; $c \neq 0 \Rightarrow$

$$\Theta_{A,\lambda} \Big|_{\frac{r}{2}}(\gamma) = \frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{2}} \sqrt{\det(A)}} \cdot$$

$$\sum_{\lambda' \in A^{-1}\mathbb{Z}^r / \mathbb{Z}^r}$$

$$\exp(-2\pi i [bB(\lambda, \lambda') + bdQ(\lambda')]) \cdot$$

$$S_\gamma(\lambda + d\lambda') \cdot \Theta_{A,\lambda'}$$

$$1. S_{-\gamma} = S_{\gamma} \wedge \left| \frac{r}{2}(-\gamma) \right| = (-1)^{\frac{r}{2}} \left| \frac{r}{2}(\gamma) \right| \Rightarrow \gamma - > -\gamma \Rightarrow \mathfrak{E} \quad c > 0$$

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2. $V_c := \text{diag}(c, 1) \Rightarrow \gamma = \begin{pmatrix} c^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & bc \\ 1 & d \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} = V_c^{-1} T^a S T^d V_c$

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$$(\Theta_{A,\lambda}|_{\frac{r}{2}}(V_c^{-1}))(\tau) = \frac{(\det(V_c^{-1}))^{\frac{r}{4}}}{1^k} \Theta_{A,\lambda}\left(\frac{\tau}{c}\right)$$

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 &= c^{-\frac{r}{4}} \sum_{\mu \in \mathbb{Z}_c^r} \left(\Theta_{cA, \frac{\lambda + \mu}{c}} \right) (\tau)
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$$\exp(\dots) \frac{(-i)^{\frac{r}{2}}}{\sqrt{\det(cA)}} \sum_{\nu \in (cA)^{-1}\mathbb{Z}^r / \mathbb{Z}^r} \exp(2\pi i B(\lambda + \mu, \nu)) \Theta_{cA, \nu}$$

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& \frac{(-i)^{\frac{r}{2}}}{\sqrt{\det(cA)}} \sum_{\nu \in (cA)^{-1} \mathbb{Z}^r / \mathbb{Z}^r} \exp \left(2\pi i \left[\frac{a}{c} Q(\lambda + \mu) + B(\lambda + \mu, \nu) + cdQ(\nu) \right] \right) \Theta_{cA, \nu}
\end{aligned}$$

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$$\Theta_{cA,\lambda} \Big|_{\frac{r}{2}} \left(V_c^{-1} T^a S T^d \right)$$

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$$\Theta_{cA, \lambda} \Big|_{\frac{r}{2}} \left(V_c^{-1} T^a S T^d \right) \stackrel{3}{=} \frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{4}} \sqrt{\det(cA)}} \cdot \sum_{\substack{\mu \in \mathbb{Z}_c^r \\ \nu \in (cA)^{-1} \mathbb{Z}^r / \mathbb{Z}^r}} \exp \left(2\pi i \left[\frac{a}{c} Q(\lambda + \mu) + B(\lambda + \mu, \nu) + cdQ(\nu) \right] \right) \Theta_{cA, \nu}$$

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8. $\neg \forall \nu \in (cA)^{-1} \mathbb{Z}^r / \mathbb{Z}^r : \exists \lambda' \in A^{-1} \mathbb{Z}^r / \mathbb{Z}^r : \exists \mu' \in \mathbb{Z}_c^r : \nu = \frac{\lambda' + \mu'}{c} \Rightarrow$

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8. $7 \wedge \forall \nu \in (cA)^{-1} \mathbb{Z}^r / \mathbb{Z}^r : \exists \lambda' \in A^{-1} \mathbb{Z}^r / \mathbb{Z}^r : \exists \mu' \in \mathbb{Z}_c^r : \nu = \frac{\lambda' + \mu'}{c} \Rightarrow$

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$$9. \quad x := \lambda + \mu, y := \lambda' + \mu' \wedge$$

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$$B(\lambda + \mu, \lambda' + \mu') - B(\lambda, \lambda') = B(\lambda, \lambda') + B(\lambda', \mu) + B(\mu, \mu') \in \mathbb{Z} \wedge$$

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$$9. \quad x := \lambda + \mu, y := \lambda' + \mu' \wedge$$

$$\begin{aligned} \frac{a}{c}Q(x) + \frac{1}{c}B(x, y) + \frac{d}{c}Q(y) &= \frac{a}{c}Q(x + dy) - bB(x, y) - bdQ(y) \wedge \\ B(\lambda + \mu, \lambda' + \mu') - B(\lambda, \lambda') &= B(\lambda, \lambda') + B(\lambda', \mu) + B(\mu, \mu') \in \mathbb{Z} \wedge \\ Q(\lambda' + \mu') - Q(\lambda') &= B(\lambda', \mu') + Q(\mu') \in \mathbb{Z} \Rightarrow \end{aligned}$$

$$\begin{aligned} \exp \left(2\pi i \left[\frac{a}{c}Q(\lambda + \mu) + \frac{1}{c}B(\lambda + \mu, \lambda' + \mu') + \frac{d}{c}Q(\lambda' + \mu') \right] \right) &= \\ \exp \left(2\pi i \left[\frac{a}{c}Q(\lambda + \mu + d\lambda' + d\mu') - bB(\lambda, \lambda') - bdQ(\lambda') \right] \right) \end{aligned}$$

$$9. \quad x := \lambda + \mu, y := \lambda' + \mu' \wedge$$

$$\frac{a}{c}Q(x) + \frac{1}{c}B(x, y) + \frac{d}{c}Q(y) = \frac{a}{c}Q(x + dy) - bB(x, y) - bdQ(y) \wedge$$

$$B(\lambda + \mu, \lambda' + \mu') - B(\lambda, \lambda') = B(\lambda, \lambda') + B(\lambda', \mu) + B(\mu, \mu') \in \mathbb{Z} \wedge$$

$$Q(\lambda' + \mu') - Q(\lambda') = B(\lambda', \mu') + Q(\mu') \in \mathbb{Z} \Rightarrow$$

$$\exp \left(2\pi i \left[\frac{a}{c}Q(\lambda + \mu) + \frac{1}{c}B(\lambda + \mu, \lambda' + \mu') + \frac{d}{c}Q(\lambda' + \mu') \right] \right) =$$

$$\exp \left(2\pi i \left[\frac{a}{c}Q(\lambda + \mu + d\lambda' + d\mu') - bB(\lambda, \lambda') - bdQ(\lambda') \right] \right)$$

$$10. \quad 8 \wedge 9 \wedge \mu \mapsto \mu - d\mu' \Rightarrow$$

$$9. \quad x := \lambda + \mu, y := \lambda' + \mu' \wedge$$

$$\frac{a}{c}Q(x) + \frac{1}{c}B(x, y) + \frac{d}{c}Q(y) = \frac{a}{c}Q(x + dy) - bB(x, y) - bdQ(y) \wedge$$

$$B(\lambda + \mu, \lambda' + \mu') - B(\lambda, \lambda') = B(\lambda, \lambda') + B(\lambda', \mu) + B(\mu, \mu') \in \mathbb{Z} \wedge$$

$$Q(\lambda' + \mu') - Q(\lambda') = B(\lambda', \mu') + Q(\mu') \in \mathbb{Z} \Rightarrow$$

$$\exp \left(2\pi i \left[\frac{a}{c}Q(\lambda + \mu) + \frac{1}{c}B(\lambda + \mu, \lambda' + \mu') + \frac{d}{c}Q(\lambda' + \mu') \right] \right) =$$

$$\exp \left(2\pi i \left[\frac{a}{c}Q(\lambda + \mu + d\lambda' + d\mu') - bB(\lambda, \lambda') - bdQ(\lambda') \right] \right)$$

$$10. \quad 8 \wedge 9 \wedge \mu \mapsto \mu - d\mu' \Rightarrow$$

$$\Theta_{cA, \lambda} \Big|_{\frac{r}{2}} \left(V_c^{-1} T^a S T^d \right)$$

$$9. \quad x := \lambda + \mu, y := \lambda' + \mu' \wedge$$

$$\frac{a}{c}Q(x) + \frac{1}{c}B(x, y) + \frac{d}{c}Q(y) = \frac{a}{c}Q(x + dy) - bB(x, y) - bdQ(y) \wedge$$

$$B(\lambda + \mu, \lambda' + \mu') - B(\lambda, \lambda') = B(\lambda, \lambda') + B(\lambda', \mu) + B(\mu, \mu') \in \mathbb{Z} \wedge$$

$$Q(\lambda' + \mu') - Q(\lambda') = B(\lambda', \mu') + Q(\mu') \in \mathbb{Z} \Rightarrow$$

$$\exp \left(2\pi i \left[\frac{a}{c}Q(\lambda + \mu) + \frac{1}{c}B(\lambda + \mu, \lambda' + \mu') + \frac{d}{c}Q(\lambda' + \mu') \right] \right) =$$

$$\exp \left(2\pi i \left[\frac{a}{c}Q(\lambda + \mu + d\lambda' + d\mu') - bB(\lambda, \lambda') - bdQ(\lambda') \right] \right)$$

$$10. \quad 8 \wedge 9 \wedge \mu \mapsto \mu - d\mu' \Rightarrow$$

$$\Theta_{cA, \lambda} \Big|_{\frac{r}{2}} \left(V_c^{-1} T^a S T^d \right) = \frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{4}} \sqrt{\det(cA)}} \cdot \sum_{\substack{\mu, \mu' \in \mathbb{Z}_c^r \\ \lambda' \in A^{-1}\mathbb{Z}^r / \mathbb{Z}^r}}$$

$$\exp \left(2\pi i \left[\frac{a}{c}Q(\lambda + \mu + d\lambda') - bB(\lambda, \lambda') - bdQ(\lambda') \right] \right) \Theta_{cA, \frac{\lambda' + \mu'}{c}}$$

11.

$$\Theta_{cA, \lambda} \Big|_{\frac{r}{2}} \left(V_c^{-1} T^a S T^d \right)$$

11.

$$\begin{aligned}
& \Theta_{cA, \lambda} \Big|_{\frac{r}{2}} \left(V_c^{-1} T^a S T^d \right) \\
& \stackrel{10}{=} \frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{4}} \sqrt{\det(cA)}} \frac{c^{-\frac{r}{4}}}{c^{-\frac{r}{4}}} \sum_{\substack{\mu \in \mathbb{Z}_c^r \\ \lambda' \in A^{-1}\mathbb{Z}^r / \mathbb{Z}^r}} \\
& \exp \left(2\pi i \left[\frac{a}{c} Q(\lambda + \mu + d\lambda') - bB(\lambda, \lambda') - bdQ(\lambda') \right] \right) \sum_{\mu' \in \mathbb{Z}_c^r} \Theta_{cA, \frac{\lambda' + \mu'}{c}}
\end{aligned}$$

11.

$$\begin{aligned}
& \Theta_{cA, \lambda} \Big|_{\frac{r}{2}} \left(V_c^{-1} T^a S T^d \right) \\
& \stackrel{10}{=} \frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{4}} \sqrt{\det(cA)}} \frac{c^{-\frac{r}{4}}}{c^{-\frac{r}{4}}} \sum_{\substack{\mu \in \mathbb{Z}_c^r \\ \lambda' \in A^{-1}\mathbb{Z}^r / \mathbb{Z}^r}} \\
& \exp \left(2\pi i \left[\frac{a}{c} Q(\lambda + \mu + d\lambda') - bB(\lambda, \lambda') - bdQ(\lambda') \right] \right) \sum_{\mu' \in \mathbb{Z}_c^r} \Theta_{cA, \frac{\lambda' + \mu'}{c}} \\
& \stackrel{3}{=} \frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{2}} \sqrt{\det(A)}} \sum_{\substack{\mu \in \mathbb{Z}_c^r \\ \lambda' \in A^{-1}\mathbb{Z}^r / \mathbb{Z}^r}} \\
& \exp \left(2\pi i \left[\frac{a}{c} Q(\lambda + \mu + d\lambda') - bB(\lambda, \lambda') - bdQ(\lambda') \right] \right) \Theta_{A, \lambda'} \Big|_{\frac{r}{2}} \left(V_c^{-1} \right)
\end{aligned}$$

12.

$$\Theta_{cA,\lambda}|_{\frac{r}{2}} \left(V_c^{-1} T^a S T^d \right)$$

12.

$$\begin{aligned} & \Theta_{cA, \lambda} \Big|_{\frac{r}{2}} \left(V_c^{-1} T^a S T^d \right) \\ & \stackrel{11}{=} \frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{2}} \sqrt{\det(A)}} \sum_{\lambda' \in A^{-1}\mathbb{Z}^r / \mathbb{Z}^r} \exp(-2\pi i [bB(\lambda, \lambda') + bdQ(\lambda')]) \cdot \\ & \quad \left[\sum_{\mu \in \mathbb{Z}_c^r} \exp\left(\frac{a}{c} Q(\mu + \lambda + d\lambda')\right) \right] \Theta_{A, \lambda'} \Big|_{\frac{r}{2}} \left(V_c^{-1} \right) \end{aligned}$$

12.

$$\begin{aligned}
& \Theta_{cA, \lambda} \Big|_{\frac{r}{2}} \left(V_c^{-1} T^a S T^d \right) \\
& \stackrel{11}{=} \frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{2}} \sqrt{\det(A)}} \sum_{\lambda' \in A^{-1}\mathbb{Z}^r / \mathbb{Z}^r} \exp(-2\pi i [bB(\lambda, \lambda') + bdQ(\lambda')]) \cdot \\
& \quad \left[\sum_{\mu \in \mathbb{Z}_c^r} \exp\left(\frac{a}{c} Q(\mu + \lambda + d\lambda')\right) \right] \Theta_{A, \lambda'} \Big|_{\frac{r}{2}} \left(V_c^{-1} \right) \\
& = \frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{2}} \sqrt{\det(A)}} \sum_{\lambda' \in A^{-1}\mathbb{Z}^r / \mathbb{Z}^r} \exp(-2\pi i [bB(\lambda, \lambda') + bdQ(\lambda')]) \cdot \\
& \quad S_\gamma(\lambda + d\lambda') \Theta_{A, \lambda'} \Big|_{\frac{r}{2}} \left(V_c^{-1} \right)
\end{aligned}$$

12.

$$\begin{aligned}
& \Theta_{cA, \lambda} \Big|_{\frac{r}{2}} \left(V_c^{-1} T^a S T^d \right) \\
& \stackrel{11}{=} \frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{2}} \sqrt{\det(A)}} \sum_{\lambda' \in A^{-1} \mathbb{Z}^r / \mathbb{Z}^r} \exp(-2\pi i [bB(\lambda, \lambda') + bdQ(\lambda')]) \cdot \\
& \quad \left[\sum_{\mu \in \mathbb{Z}_c^r} \exp\left(\frac{a}{c} Q(\mu + \lambda + d\lambda')\right) \right] \Theta_{A, \lambda'} \Big|_{\frac{r}{2}} \left(V_c^{-1} \right) \\
& = \frac{(-i)^{\frac{r}{2}}}{c^{\frac{r}{2}} \sqrt{\det(A)}} \sum_{\lambda' \in A^{-1} \mathbb{Z}^r / \mathbb{Z}^r} \exp(-2\pi i [bB(\lambda, \lambda') + bdQ(\lambda')]) \cdot \\
& \quad S_\gamma(\lambda + d\lambda') \Theta_{A, \lambda'} \Big|_{\frac{r}{2}} \left(V_c^{-1} \right)
\end{aligned}$$

$$13. \quad 2, 11 \Rightarrow \Theta_{A, \lambda} \Big|_{\frac{r}{2}}(\gamma) = \Theta_{cA, \frac{\lambda + \mu}{c}} \Big|_{\frac{r}{2}} \left(V_c^{-1} T^a S T^d V_c \right)$$

□

Ende

Vielen Dank für Ihre Aufmerksamkeit!